Lambert's Problem Revisited

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Two classical forms of the time of flight equation for the two body, two point boundary value problem, known as "Lambert's problem," are combined to produce an elegant formulation which may serve as the nucleus of an extremely efficient computation algorithm. The time equation is universal (i.e., includes elliptic, parabolic, and hyperbolic orbits), is a well-behaved function of a single convenient independent variable, and requires the evaluation of a single hypergeometric function. Recursive identities for hypergeometric functions are developed and effectively used to enhance computation speed. A "top-down" continued fraction algorithm, used for efficient evaluation of the hypergeometric function, is also exploited. This formulation represents a considerable improvement over other methods known to the author.

Introduction

THE determination of an orbit, having a specified flight time and connecting two position vectors, frequently referred to as Lambert's problem, is fundamental as a means of targeting spacecraft and missiles. A variety of methods for dealing with this problem has been discussed by many writers. (For a fairly comprehensive list of references, see Escobal,[1] Herrick,[2] and Battin.[3,4]) Each method is characterized primarily by a particular form of the time of flight equation and a particular independent variable to be used in an iteration algorithm to determine the orbital elements.

It is the purpose of this paper to present an extremely compact and computationally efficient time of flight expression (encompassing elliptic, parabolic and hyperbolic orbits) as a well-behaved function of a single, physically significant, independent variable. The calculation involves the evaluation of only one hypergeometric function expressed as a continued fraction and, for many applications, as few as one square root function per iteration cycle. In an earlier paper by the present author,[4] it was required that two hypergometric functions be computed per cycle.

This formulation of the time equation is in terms of the same independent variable x as used by Battin[4] and Lancaster and Blanchard.[5] Specifically, x is defined by

$$x^2 = 1 - a_m/a$$

where a and a_m are, respectively, the semimajor axes of the desired orbit and the minimum energy orbit which fulfill the terminal boundary conditions. All elliptic orbits correspond to the range $-1<x<1$, hyperbolic orbits to $x>1$, and the parabolic orbit to $x=1$. It was shown in Battin[4] that x is proportional at either terminal to a particular component of the velocity vector when expressed in a specific nonsingular local orthogonal coordinate system.

A significant portion of the paper is devoted to developing and applying some recursive identities for hypergeometric functions which can be used to improve the computational efficiency of the time of flight equation. Of further interest is a continued fraction calculation algorithm which permits evaluation of the fraction from "the top to the bottom" without the necessity of repeated scaling.

The Lambert problem formulation described here represents a considerable improvement over any of the various methods known to the author, particularly when used as an integral part of a powered flight guidance algorithm, for which the orbit determination must be made in real time. Then it is essential that the time of flight equation, which must be solved iteratively, be as simple to evaluate as possible.

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Time of Flight Equations

In this section we derive two separate time of flight relationships, the first expressing the flight time as a function of the difference of the eccentric anomalies at the two termini and the second as a function of the variable x defined in the Introduction. Then, we combine the best features of these two formulations in developing the final results.

To begin, we summarize, using a standard notation, the basic equations of two body elliptic motion:

$$\sqrt{\mu}(t-\tau) = a^{3/2}(E - e\sin E) \qquad \text{Kepler's Eq.}$$

$$r = a(1 - e\cos E)$$

$$r\cos f = a(\cos E - e), \qquad r\sin f = a\sqrt{1-e^2}\sin E$$

$$\sqrt{r}\cos{\tfrac{1}{2}}f = \sqrt{a(1-e)}\cos{\tfrac{1}{2}}E, \quad \sqrt{r}\sin{\tfrac{1}{2}}f = \sqrt{a(1+e)}\sin{\tfrac{1}{2}}E$$

For the geometry of the boundary value problem, let the initial and target points be P_1 and P_2 having radial distances r_1 and r_2 from the force center as shown in Fig. 1. Further, define c as the line-of-sight distance from P_1 to P_2. Denote by s the semiperimeter of the triangle whose sides are r_1, r_2, c so that the semimajor axis a_m of the minimum energy orbit connecting P_1 and P_2 is

$$2\,a_m = s = {\tfrac{1}{2}}(r_1 + r_2 + c) \tag{1}$$

Also, for notational convenience, we introduce the parameter λ defined as

$$\lambda s = \sqrt{r_1 r_2}\cos\,{\tfrac{1}{2}}(f_2 - f_1) \tag{2}$$

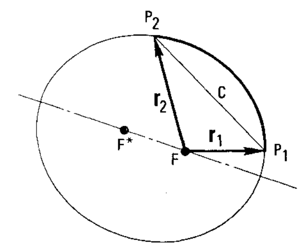

Fig. 1 Geometry of boundary-value problem.

Since

$$\lambda^2 = (s-c)/s \tag{3}$$

then λ has a range $(-1, 1)$ and, of course, depends only on the geometrical configuration of P_1 and P_2 with respect to the attractive focus.

Now, with E_1 and E_2 denoting the eccentric anomalies at P_1 and P_2, we define two quantities ψ and ϕ by

$$\psi = \tfrac{1}{2}(E_2 - E_1), \quad \cos\phi = e\cos\tfrac{1}{2}(E_2 + E_1) \tag{4a}$$

so that we may write

$$\sqrt{\mu}(t_2 - t_1) = 2a^{3/2}(\psi - \sin\psi\,\cos\phi) \tag{5}$$

$$r_1 + r_2 = 2a(1 - \cos\psi\,\cos\phi) \tag{6}$$

$$c = 2a\sin\psi\,\sin\phi \tag{7}$$

$$\lambda s = a(\cos\psi - \cos\phi) \tag{8}$$

Similar equations obtain for two body hyperbolic motion using the basic equations

$$\sqrt{\mu}(t-\tau) = (-a)^{3/2}(e\sinh H - H)$$

$$r = a(1 - e\cosh H)$$

with ψ and ϕ now defined as

$$\psi = \tfrac{1}{2}(H_2 - H_1), \quad \cosh\phi = e\cosh\tfrac{1}{2}(H_2 + H_1) \tag{4b}$$

Gauss' Equation

If we eliminate $\cos\phi$ between Eqs. (5) and (8), the time of flight $t_2 - t_1$ for an elliptic orbit connecting P_1 and P_2 may be written as

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}(2\psi - \sin 2\psi) + 2\lambda s a^{1/2}\sin\psi \tag{9a}$$

as first obtained by Gauss in his Theoria Motus.

Similarly, for hyperbolic orbits, we have

$$\sqrt{\mu}(t_2 - t_1) = (-a)^{3/2}(\sinh 2\psi - 2\psi) + 2\lambda s(-a)^{1/2}\sinh\psi \tag{9b}$$

We now define a positive quantity η by

$$s\eta^2 = \begin{cases} 2a\sin^2\psi & (10a) \\ -2a\sinh^2\psi & (10b) \end{cases}$$

and a function Q_ψ as

$$Q_\psi = \begin{cases} \dfrac{2\psi - \sin 2\psi}{\sin^3\psi} & (11a) \\[2ex] \dfrac{\sinh 2\psi - 2\psi}{\sinh^3\psi} & (11b) \end{cases}$$

so that the time of flight equation, in either case, may be written as

$$\sqrt{\mu/a_m^{\,3}}(t_2 - t_1) = \eta^3 Q_\psi + 4\lambda\eta \tag{12}$$

Finally, by eliminating ϕ between Eqs. (6) and (8), we have

$$r_1 + r_2 = s\eta^2 + 2\lambda s\begin{cases}\cos\psi \\ \cosh\psi\end{cases}$$

so that for fixed geometry, η is a function only of ψ. Indeed, since

$$r_1 + r_2 = 2s - c = s(1 + \lambda^2)$$

we have

$$\eta^2 = \begin{cases} (1-\lambda)^2 + 4\lambda\sin^2\tfrac{1}{2}\psi & (13a) \\ (1-\lambda)^2 - 4\lambda\sinh^2\tfrac{1}{2}\psi & (13b) \end{cases}$$

The time of flight is thus a function only of the anomaly difference $E_2 - E_1$ (or $H_2 - H_1$). However, since (as shown in the following section)

$$Q_\psi = \begin{cases} \dfrac{4}{3}F(3,1;\dfrac{5}{2};\sin^2\tfrac{1}{2}\psi) & (14a) \\[2ex] \dfrac{4}{3}F(3,1;\dfrac{5}{2};-\sinh^2\tfrac{1}{2}\psi) & (14b) \end{cases}$$

where F is a hypergeometric function, we may define

$$S_1 = \begin{cases} \sin^2\tfrac{1}{2}\psi & (15a) \\ -\sinh^2\tfrac{1}{2}\psi & (15b) \end{cases}$$

and express $t_2 - t_1$ more compactly as a function of S_1. Thus

$$\sqrt{\frac{\mu}{a_m^3}}(t_2 - t_1) = \frac{4}{3}\eta^3 F(3, 1; \frac{5}{2}; S_1) + 4\lambda\eta \tag{16}$$

$$\eta^2 = (1-\lambda)^2 + 4\lambda S_1, \quad \eta \geq 0 \tag{17}$$

where

$0 < S_1 < 1$	elliptic orbits
$S_1 = 0$	parabolic orbit
$-\infty < S_1 < 0$	hyperbolic orbits

A graph of the time of flight as a function of $-S_1$ for various values of λ is given in Fig. 2.

Lagrange's Equation

If we define two quantities α and β as

$$\alpha = \phi + \psi, \quad \beta = \phi - \psi \tag{18}$$

so that

$$\psi = \tfrac{1}{2}(\alpha - \beta), \quad \phi = \tfrac{1}{2}(\alpha + \beta) \tag{19}$$

then, from Eqs. (6) and (7), we have

$$2s = r_1 + r_2 + c = 2a(1 - \cos\alpha)$$

$$2(s-c) = r_1 + r_2 - c = 2a(1 - \cos\beta)$$

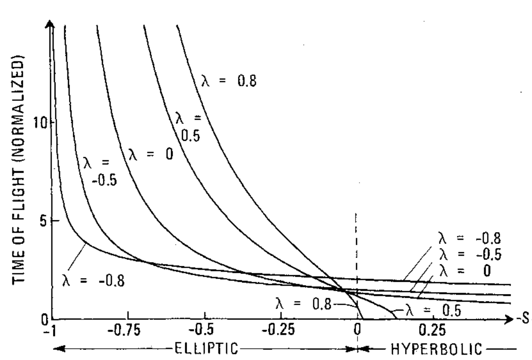

Fig. 2 Time of flight as a function of $-S_1$.

or simply

$$\sin^2 \tfrac{1}{2}\alpha = s/2a, \qquad \sin^2 \tfrac{1}{2}\beta = (s-c)/2a \tag{20a}$$

Also, since Eq. (8) may be written as

$$\lambda s = 2a \sin \tfrac{1}{2}\alpha \ \sin \tfrac{1}{2}\beta \tag{21}$$

we may combine this with the first of Eqs. (20a) and obtain the following relation between α and β

$$\sin \tfrac{1}{2}\beta = \lambda \sin \tfrac{1}{2}\alpha \tag{22a}$$

The time of flight may be expressed in terms of α and β by substituting Eqs. (19) into Eq. (5). We have

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}[(\alpha - \sin\alpha) - (\beta - \sin\beta)] \tag{23}$$

as first derived by Lagrange. Note that $(t_2 - t_1)$, for fixed geometry, is a function only of the semimajor axis a or, alternatively, the quantity α. It is shown in Battin[3,4] that Eq. (23) is valid for all possible elliptic transfers $(f_2 - f_1 < 2\pi)$ with the convention

$$0 \le \alpha \le 2\pi, \quad 0 \le \beta \le \pi \ \text{ for } f_2 - f_1 \le \pi$$

$$0 \le \alpha \le 2\pi, \quad -\pi \le \beta \le 0 \ \text{ for } f_2 - f_1 \ge \pi$$

Finally, we use Eqs. (20a) to write

$$\sqrt{\mu/a_m{}^3}\,(t_2 - t_1) = Q_{\frac{1}{2}\alpha} - \lambda^3 Q_{\frac{1}{2}\beta} \tag{24}$$

where the function $Q_{(\ )}$ is defined in Eq. (11a).

The same time of flight equation [Eq. (24)] is obtained for hyperbolic orbits with α and β also defined in terms of ψ and ϕ through Eqs. (18) and (19). In this case, the relation to the semimajor axis $(-a)$ is according to

$$\sinh^2 \tfrac{1}{2}\alpha = s/(-2a), \ \sinh^2 \tfrac{1}{2}\beta = (s-c)/(-2a) \tag{20b}$$

with α and β related by

$$\sinh \tfrac{1}{2}\beta = \lambda \ \sinh \tfrac{1}{2}\alpha \tag{22b}$$

and the following sign convention invoked

$$0 \le \alpha, \ 0 \le \beta \ \text{ for } f_2 - f_1 \le \pi$$

$$0 \le \alpha, \ \beta \le 0 \ \text{ for } f_2 - f_1 \ge \pi$$

The advantage of expressing the time of flight as a function of the variable x, where

$$x = \begin{cases} \cos \tfrac{1}{2}\alpha & (25a) \\ \cosh \tfrac{1}{2}\alpha & (25b) \end{cases}$$

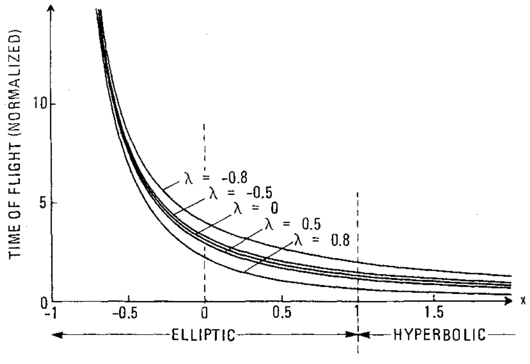

Fig. 3 Time of flight as a function of x.

was fully explored by Battin[4] and Lancaster and Blanchard.[5] In terms of x, and with the positive quantity y defined as

$$y = \begin{cases} \cos \tfrac{1}{2}\beta & (26a) \\ \cosh \tfrac{1}{2}\beta & (26b) \end{cases}$$

and related to x by

$$y = \sqrt{1 - \lambda^2 (1 - x^2)} \tag{27}$$

we may write Eq. (24) (as shown in the following section) in the form

$$\sqrt{\frac{\mu}{a_m{}^3}}\,(t_2 - t_1) = \frac{4}{3} F(3,\ 1;\ \frac{5}{2};\ \frac{1-x}{2}) - \frac{4}{3} \lambda^3 F(3,\ 1;\ \frac{5}{2};\ \frac{1-y}{2}) \tag{28}$$

A graph of the time of flight vs x for various values of λ is given in Fig. 3. We note that

$-1 < x < 1$	elliptic orbits
$x = 1$	parabolic orbit
$1 < x < \infty$	hyperbolic orbits

and also that $x = 0$ corresponds to the time of flight for the minimum energy path from P_1 to P_2.

Combined Equations

When we compare the two time of flight formulations, as summarized in Figs. 2 and 3, it appears that the variation of $t_2 - t_1$ as a function of x is more amenable to a mechanized iterative solution than is its variation as a function of S_1. The two sets of curves are identical for $\lambda = 0$, corresponding to a 180° transfer, but otherwise differ significantly in important characteristics.

On the other hand, if the basis of comparison is computational efficiency, then Gauss' equation is to be preferred since the evaluation of only one hypergeometric function is required rather than two. In both cases, one square root function is necessary.

It is possible to relate the independent variables S_1 and x in a simple way so that the advantages of both formulations can be realized in a single expression. For this purpose, using Eqs. (15) and (19), we write

$$2S_1 = \begin{cases} 1 - \cos\psi = 1 - \cos \tfrac{1}{2}(\alpha - \beta) \\ 1 - \cosh\psi = 1 - \cosh \tfrac{1}{2}(\alpha - \beta) \end{cases}$$

$$= \begin{cases} 1 - \cos \tfrac{1}{2}\alpha \ \cos \tfrac{1}{2}\beta - \sin \tfrac{1}{2}\alpha \ \sin \tfrac{1}{2}\beta \\ 1 - \cosh \tfrac{1}{2}\alpha \ \cosh \tfrac{1}{2}\beta + \sinh \tfrac{1}{2}\alpha \ \sinh \tfrac{1}{2}\beta \end{cases}$$

Next, we employ Eqs. (22a) and (22b) to eliminate $\sin \tfrac{1}{2}\beta$ and $\sinh \tfrac{1}{2}\beta$. There results

$$2S_1 = \begin{cases} 1 - \cos \tfrac{1}{2}\alpha \ \cos \tfrac{1}{2}\beta - \lambda \sin^2 \tfrac{1}{2}\alpha \\ 1 - \cosh \tfrac{1}{2}\alpha \ \cosh \tfrac{1}{2}\beta + \lambda \sinh^2 \tfrac{1}{2}\alpha \end{cases}$$

$$= \quad 1 - xy - \lambda(1 - x^2) \tag{29}$$

Furthermore, the quantity η^2, as given in Eq. (13), is a much simpler expression in terms of x and y. Thus

$$\eta^2 = \begin{cases} 1 + \lambda^2 - 2\lambda \cos \tfrac{1}{2}(\alpha - \beta) \\ 1 + \lambda^2 - 2\lambda \cosh \tfrac{1}{2}(\alpha - \beta) \end{cases}$$

which we expand, using Eqs. (22a) and (22b), as before to obtain

$$\eta^2 = \begin{cases} 1-\lambda^2-2\lambda\cos{\tfrac{1}{2}}\alpha\ \cos{\tfrac{1}{2}}\beta+2\lambda^2\cos^2{\tfrac{1}{2}}\alpha \\ 1-\lambda^2-2\lambda\cosh{\tfrac{1}{2}}\alpha\ \cosh{\tfrac{1}{2}}\beta+2\lambda^2\cosh^2{\tfrac{1}{2}}\alpha \end{cases}$$

$$= \quad 1-\lambda^2-2\lambda xy+2\lambda^2x^2$$

Then, from Eq. (27)

$$1-\lambda^2 = y^2-\lambda^2x^2$$

so that

$$\eta^2 = (y-\lambda x)^2$$

Finally, since y and η are both positive and $y^2-\lambda^2x^2 \geq 0$, it follows that

$$\eta = y-\lambda x \tag{30}$$

Therefore, the time of flight formulation, which seems more appropriate then either Gauss' or Lagrange's alone, can be summarized as

$$2a_m = s = {\tfrac{1}{2}}(r_1+r_2+c) \tag{1}$$

$$\lambda s = \sqrt{r_1r_2}\cos{\tfrac{1}{2}}(f_2-f_1) \tag{2}$$

$$y = \sqrt{1-\lambda^2(1-x^2)} \tag{27}$$

$$\eta = y-\lambda x \tag{30}$$

$$S_1 = {\tfrac{1}{2}}(1-\lambda-x\eta) \tag{29}$$

$$Q = \frac{4}{3}F(3,\ 1;\ \frac{5}{2};\ S_1) \tag{14}$$

$$\sqrt{\mu/a_m{}^3}(t_2-t_1) = \eta^3Q+4\lambda\eta \tag{12†}$$

where x is to be regarded as the independent or iterated variable. One square root and one hypergeometric function are required for each computation cycle.

In the Appendix convenient derivative formulas for the time of flight are developed for the case in which a Newton-Raphson method is to be used to find x.

The Q Function

The function Q_ψ, or simply Q, defined in Eq. (11a) as

$$Q = (2\psi-\sin 2\psi)/\sin^3\psi$$

can be shown to be a hypergeometric function of

$$z = \sin^2{\tfrac{1}{2}}\psi$$

To this end, we differentiate Q with respect to z to obtain

$$\sin^2\psi\frac{dQ}{dz}+6\ \cos\psi\ Q = 8$$

By substituting for ψ

$$\sin^2\psi = 4z(1-z), \quad \cos\psi = 1-2z$$

and differentiating a second time, we obtain

$$z(1-z)\frac{d^2Q}{dz^2}+(\frac{5}{2}-5z)\frac{dQ}{dz}-3Q = 0$$

†For elliptic orbits, if we want to include the possibility of a number N of complete orbits before termination at point P_2, Eq. (12) is modified by adding $2N\pi\ (1-x^2)^{-3/2}$ to the right-hand side (Battin[4]).

This is Gauss' equation with parameters 3, 1, and 5/2. Since

$$\lim_{z\to 0} Q = \lim_{\psi\to 0} Q = \frac{4}{3}$$

then, Eq. (14a) of the previous section is established and, in a similar manner, we can verify Eq. (14b).

The chief advantage in formulating the time of flight equation in terms of hypergeometric functions is that a wide variety of possibilities for computational efficiency can be explored in an orderly fashion. For this purpose, we will develop some convenient recursive relations which are direct consequences of the following general properties of hypergeometric functions[6]:

1) Gauss' relation for contiguous functions

$$(c-a-b)F(a,b;\ c;\ z)+a(1-z)F(a+1,\ b;\ c;\ z)$$
$$-(c-b)F(a,\ b-1;\ c;\ z) = 0$$

2) Linear transformation formula

$$F(a,\ b;\ c;\ z) = (1-z)^{-a}F(a,\ c-b;\ c;\frac{z}{z-1})$$

3) Quadratic transformation formula

$$F(a,\ b;\ a+b-{\tfrac{1}{2}};\ z)$$
$$= (1-z)^{-1/2}F(2a-1,\ 2b-1;\ a+b-{\tfrac{1}{2}};\ {\tfrac{1}{2}}-{\tfrac{1}{2}}\sqrt{1-z})$$

For notational convenience, we define

$$F_1(z) = F(3,\ 1;\ \frac{5}{2};\ z)$$

$$F_2(z) = F(-\frac{1}{2},\ 1;\ \frac{5}{2};\ z)$$

The z arguments in the identities to be derived may be either circular or hyperbolic functions so that we shall also define the functions

$$S_n = \begin{cases} \sin^2\dfrac{\psi}{2^n} \\ -\sinh^2\dfrac{\psi}{2^n} \end{cases} \quad C_n = \begin{cases} \cos^2\dfrac{\psi}{2^n} \\ \cosh^2\dfrac{\psi}{2^n} \end{cases} \quad T_n = \begin{cases} -\tan^2\dfrac{\psi}{2^n} \\ \tanh^2\dfrac{\psi}{2^n} \end{cases} \tag{31}$$

for which the following recursive relations hold with $n=0,\ 1,\ 2,\ldots$

$$S_{n+1} = {\tfrac{1}{2}}(1-\sqrt{C_n}) \tag{32a}$$

$$C_{n+1} = {\tfrac{1}{2}}(1+\sqrt{C_n}) \tag{32b}$$

$$T_{n+1} = -\frac{S_{n+1}}{C_{n+1}} = \frac{\sqrt{C_n}-1}{\sqrt{C_n}+1} \tag{32c}$$

The first recursive expression

$$F_1(S_n) = \frac{1}{4C_n}\left[3+\frac{1}{\sqrt{C_n}}F_1(S_{n+1})\right] \tag{33}$$

is established from the relation for contiguous functions and the quadratic transformation formula by setting $a=2$, $b=1$, $c=5/2$ and $z=S_n$. The identity

$$F_1(S_n) = \frac{1}{C_n}F_2(T_n) \tag{34}$$

follows at once from the linear transformation formula with $a=1$, $b=3$, $c=5/2$ and $z=S_n$. Finally, the second recursive expression

$$F_2(T_n)=\frac{1}{4}\left[3+\frac{1}{C_{n+1}\sqrt{C_n}}F_2(T_{n+1})\right] \tag{35}$$

is obtained by combining Eqs. (33) and (34).

In terms of this notation, the Q function may be written as

$$Q=\frac{4}{3}F_1(S_1) \tag{36}$$

or

$$Q=\frac{4}{3C_1}F_2(T_1) \tag{37}$$

where S_1 is defined in Eq. (29) and

$$C_1=\tfrac{1}{2}(1+\lambda+x\eta) \tag{38}$$

$$T_1=\frac{\lambda+x\eta-1}{\lambda+x\eta+1} \tag{39}$$

Then, since convergence of the hypergeometric series is enhanced when the argument is small, we may recursively use Eqs. (33) and (35) to advantage in order to obtain, for given ψ, as rapid convergence properties as desired. There is, of course, a penalty in that the expression for Q becomes algebraically more complex. For example, if we apply the recursive identities successively, we generate the following two sequences for Q

$$Q=\frac{4}{3}F_1(S_1)$$

$$Q=\frac{1}{C_1}\left[1+\frac{1}{3\sqrt{C_1}}F_1(S_2)\right]$$

$$Q=\frac{1}{C_1}\left\{1+\frac{1}{4C_2\sqrt{C_1}}\left[1+\frac{1}{3\sqrt{C_2}}F_1(S_3)\right]\right\}\text{etc.}$$

where

$$S_1=\tfrac{1}{2}(1-\lambda-x\eta) \qquad C_1=\tfrac{1}{2}(1+\lambda+x\eta)$$

$$S_2=\tfrac{1}{2}(1-\sqrt{C_1}) \qquad C_2=\tfrac{1}{2}(1+\sqrt{C_1})$$

$$S_3=\tfrac{1}{2}(1-\sqrt{C_2})$$

and

$$Q=\frac{4}{3C_1}F_2(T_1)$$

$$Q=\frac{1}{C_1}\left[1+\frac{1}{3C_2\sqrt{C_1}}F_2(T_2)\right]$$

$$Q=\frac{1}{C_1}\left\{1+\frac{1}{4C_2\sqrt{C_1}}\left[1+\frac{1}{3C_3\sqrt{C_2}}F_2(T_3)\right]\right\}\text{etc.}$$

where

$$T_1=\frac{\lambda+x\eta-1}{\lambda+x\eta+1} \qquad C_1=\tfrac{1}{2}(1+\lambda+x\eta)$$

$$T_2=\frac{\sqrt{C_1}-1}{\sqrt{C_1}+1} \qquad C_2=\tfrac{1}{2}(1+\sqrt{C_1})$$

$$C_3=\tfrac{1}{2}(1+\sqrt{C_2})$$

$$T_3=\frac{\sqrt{C_2}-1}{\sqrt{C_2}+1}$$

Note that each time the recursion is applied an additional square root is required in the calculation of Q while, at the same time, the magnitudes of the arguments of the hypergeometric functions decrease as indicated in Eqs. (31).

Rather than by power series, a considerably more efficient method of evaluating F_1 and F_2 is by means of continued fraction expansions. Gauss' theorem for the ratio of two associated hypergeometric functions[8] can be exploited to develop F_1 and F_2 as continued fractions. Thus, we may write, for $i=1,2$

$$F_i(z)=\cfrac{1}{1-\cfrac{\gamma_1^{(i)}z}{1-\cfrac{\gamma_2^{(i)}z}{1-\ddots}}} \tag{40}$$

where

$$\gamma_n^{(1)}=\begin{cases}\dfrac{(n+2)(n+5)}{(2n+1)(2n+3)} & n \text{ odd}\\[2ex] \dfrac{n(n-3)}{(2n+1)(2n+3)} & n \text{ even}\end{cases} \tag{41}$$

$$\gamma_n^{(2)}=\begin{cases}\dfrac{(n-2)(n+2)}{(2n+1)(2n+3)} & n \text{ odd}\\[2ex] \dfrac{n(n+4)}{(2n+1)(2n+3)} & n \text{ even}\end{cases} \tag{42}$$

Not only is the speed of convergence improved, for a given argument z, but the range of convergence is also expanded from $|z|<1$ for the power series to $z<1$ for the continued fraction representation.[7] Note that this latter range encompasses the entire spectrum of arguments for Lambert's problem.

A convenient technique for evaluating continued fractions from the top to the bottom was presented by Gautschi.[9] Applying this to the case at hand, the algorithm for determining $F_i(z)$ may be summarized as follows:

$$\begin{aligned}\delta_1&=u_1=\Sigma_1=1\\ \delta_{n+1}&=\frac{1}{1-\gamma_n^{(i)}z\delta_n}\\ u_{n+1}&=u_n(\delta_{n+1}-1)\\ \Sigma_{n+1}&=\Sigma_n+u_{n+1}\end{aligned} \tag{43}$$

where $\gamma_n^{(i)}$ are given in Eqs. (41) and (42). Successive calculations, for $n=1, 2, \ldots,$ of these three equations produces finally $F_i(z)$ since

$$F_i(z)=\lim_{n\to\infty}\Sigma_n \tag{44}$$

provided, of course, that $z<1$.

The convergence properties of the continued fraction expansions of F_1 and F_2 were analyzed and the results are summarized in Table 1. The number of levels in the expansion required for 10 decimal digit accuracy is tabulated for $F_1(S_n)$ with $n=1, 2, 3$ as a function of the independent variable x and the geometry parameter λ. Similar data for $F_2(T_n)$ are not presented since this function was found to have almost identical characteristics. It should further be noted that although the behavior of $F_1(S_4)$ is not explicitly given, the number of levels required is smaller by at most one or two than required for $F_1(S_3)$. Finally, for a partial analysis comparing the convergence properties of series expansions and continued fraction expansions, the reader is referred to Battin.[4]

The Velocity Vector

The final step in the solution of the Lambert boundary value problem is the calculation of the velocity vector $\boldsymbol{v}_1$ at the point P_1 in terms of that value of x which has been found to satisfy the time of flight equation [Eq. (12)].

From the basic mechanics of two body motion, the vector $\boldsymbol{v}_1$ may be written as

$$\boldsymbol{v}_1 = \frac{\sqrt{\mu}}{r_1}(\sigma_1 \boldsymbol{i}_{r_1} + \sqrt{p}\ \boldsymbol{i}_\zeta \times \boldsymbol{i}_{r_1}) \tag{45}$$

where $\boldsymbol{i}_{r_1}$ is the unit vector defining the direction of P_1 from the force center, $\boldsymbol{i}_\zeta$ the unit vector normal to the orbital plane, p the parameter of the orbit, and

$$\sigma_1 = \frac{\boldsymbol{r}_1 \cdot \boldsymbol{v}_1}{\sqrt{\mu}}$$

To complete the task, we must find convenient expressions for p and σ_1.

First, it is shown in Battin[4] that

$$p = \frac{4 r_1 r_2 a_m}{c^2}(y + \lambda x)^2 \sin^2 \tfrac{1}{2}(f_2 - f_1) \tag{46}$$

and since

$$(y + \lambda x)(y - \lambda x) = 1 - \lambda^2 = \frac{c}{s} = \eta(y + \lambda x)$$

we may also write

$$p = \frac{r_1 r_2}{a_m \eta^2}\sin^2 \tfrac{1}{2}(f_2 - f_1) \tag{47}$$

Thus, the parameter of the orbit is seen to be inversely proportional to η^2.

Second, by comparing the two forms of the classical Lagrange F and G functions,[3] one in terms of the true anomaly difference $f_2 - f_1$ and the other in terms of the eccentric anomaly difference $E_2 - E_1$ (or $H_2 - H_1$), we have the following equations for σ_1

$$\frac{\sigma_1}{\sqrt{p}}\sin \tfrac{1}{2}(f_2 - f_1) = \cos \tfrac{1}{2}(f_2 - f_1) - \sqrt{\frac{r_1}{r_2}}\begin{Bmatrix}\cos\psi \\ \cosh\psi\end{Bmatrix}$$

which, using Eqs. (47), (2), and (29), become

$$\sigma_1 = \frac{1}{\eta\sqrt{a_m}}[2\lambda a_m - r_1(\lambda + x\eta)] \tag{48}$$

Finally, then, Eq. (45) may be expressed as

$$\boldsymbol{v}_1 = \frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\left\{\left[2\lambda\frac{a_m}{r_1} - (\lambda + x\eta)\right]\boldsymbol{i}_{r_1} + \sqrt{\frac{r_2}{r_1}}\sin \tfrac{1}{2}(f_2 - f_1)\,\boldsymbol{i}_\zeta \times \boldsymbol{i}_{r_1}\right\} \tag{49}$$

which is a most convenient form for computational purposes.

A different expression for the velocity vector $\boldsymbol{v}_1$ is developed in Battin[4] which gives physical significance to the variables x and y.

Table 1 Number of continued fraction levels required to compute $F(3, 1; 5/2; S_n), n = 1, 2, 3$

Range of x: $x_0 < x < x_1$		No. of Levels for $F_1(S_1)$, $F_1(S_2)$, $F_1(S_3)$ for 10^{-10} Accuracy				
x_0	x_1	$\lambda = -0.8$	$\lambda = -0.5$	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 0.8$
- .9,	- .8	a, 13, 7	121, 12, 7	58, 11, 7	37, 10, 6	30, 10, 6
- .8,	- .7	a, 13, 7	80, 12, 7	39, 10, 6	25, 9, 6	20, 9, 6
- .7,	- .6	a, 12, 7	61, 11, 7	31, 10, 6	20, 9, 6	16, 8, 6
- .6,	- .5	120, 12, 7	50, 11, 7	26, 10, 6	16, 8, 6	13, 7, 5
- .5,	- .4	98, 12, 7	42, 11, 7	22, 9, 6	14, 8, 6	11, 7, 5
- .4,	- .3	80, 12, 7	36, 10, 6	20, 9, 6	13, 7, 5	10, 6, 5
- .3,	- .2	67, 12, 7	32, 10, 6	18, 8, 6	12, 7, 5	9, 6, 5
- .2,	- .1	56, 11, 7	28, 10, 6	16, 8, 6	10, 6, 5	8, 6, 4
- .1,	0	47, 11, 7	25, 9, 6	15, 8, 6	10, 6, 5	7, 5, 4
0,	.1	39, 10, 6	22, 9, 6	14, 8, 5	9, 6, 5	7, 5, 4
.1,	.2	33, 10, 6	20, 9, 6	12, 7, 5	8, 6, 4	6, 5, 4
.2,	.3	28, 10, 6	18, 8, 6	12, 7, 5	8, 6, 4	6, 4, 4
.3,	.4	23, 9, 6	16, 8, 6	11, 7, 5	7, 5, 4	5, 4, 4
.4,	.5	20, 9, 6	14, 8, 6	10, 6, 5	7, 5, 4	5, 4, 4
.5,	.6	17, 8, 6	13, 7, 5	9, 6, 5	6, 5, 4	5, 4, 3
.6,	.7	14, 8, 6	11, 7, 5	8, 6, 4	6, 5, 4	4, 4, 3
.7,	.8	12, 7, 5	10, 6, 5	8, 5, 4	6, 4, 4	4, 4, 3
.8,	.9	10, 6, 5	8, 6, 4	7, 5, 4	5, 4, 4	4, 3, 3
.9,	1.0	8, 6, 4	7, 5, 4	6, 4, 4	4, 4, 3	4, 3, 3
x = 1		1, 1, 1	1, 1, 1	1, 1, 1	1, 1, 1	1, 1, 1
1.0,	1.1	7, 5, 4	7, 5, 4	6, 4, 4	4, 4, 3	4, 3, 3
1.1,	1.2	8, 6, 5	8, 6, 4	6, 5, 4	5, 4, 4	4, 3, 3
1.2,	1.3	10, 7, 5	9, 6, 5	7, 5, 4	5, 4, 4	4, 3, 3
1.3,	1.4	11, 7, 5	10, 7, 5	8, 6, 4	5, 4, 4	4, 4, 3
1.4,	1.5	12, 8, 6	11, 7, 5	8, 6, 4	6, 4, 4	4, 4, 3
1.5,	1.6	13, 8, 6	12, 7, 5	8, 6, 5	6, 4, 4	4, 4, 3
1.6,	1.7	14, 8, 6	12, 8, 6	9, 6, 5	6, 4, 4	4, 4, 3
1.7,	1.8	15, 9, 6	13, 8, 6	9, 6, 5	6, 5, 4	4, 4, 3
1.8,	1.9	16, 9, 6	14, 8, 6	10, 6, 5	6, 5, 4	4, 4, 3

[a] Greater than 140 levels (roundoff prevents 10^{-10} accuracy).

Appendix

When a Newton-Raphson method is used to solve the time of flight equation for x, the derivative of Eq. (12) is required. For this purpose, we use Eqs. (27) and (30) to obtain

$$d\eta/dx = -\lambda\eta/y$$

so that we have

$$\frac{d}{dx}\sqrt{\frac{\mu}{a_m^3}}\,(t_2-t_1) = \eta^3\frac{dQ}{dx} - \frac{\lambda\eta}{y}\,(3\eta^2 Q + 4\lambda)$$

The derivative of the Q function implies differentiation of the hypergeometric functions $F_1(z)$ or $F_2(z)$. In each case, we have[6]

$$\frac{dF_1}{dz} = \frac{3(1-F_1)+6zF_1}{2z(1-z)}$$

$$\frac{dF_2}{dz} = \frac{3(1-F_2)-zF_2}{2z(1-z)}$$

Unfortunately, these expressions are both indeterminant when $z=0$ which corresponds to the case of a parabolic orbit.

To resolve the indeterminancy, we write, for $i=1, 2$

$$F_i(z) = 1/(1-\gamma^{(i)}zG_i)$$

from the continued fraction representation given in Eq. (40) where

$$G_i(z) = \cfrac{1}{1-\cfrac{\gamma_2^{(i)}z}{1-\cfrac{\gamma_3^{(i)}z}{1-\ddots}}}$$

Then we have

$$\frac{dF_1}{dz} = \frac{6-3\gamma_1^{(1)}G_1}{2(1-z)(1-\gamma_1^{(1)}zG_1)}$$

$$\frac{dF_2}{dz} = -\frac{1+3\gamma_1^{(2)}G_2}{2(1-z)(1-\gamma_1^{(2)}zG_2)}$$

The functions G_i (instead of F_i) may be evaluated by a trivial modification of the algorithm summarized in Eqs. (43).

Finally, the derivative of z with respect to x, where $z=S_n$ or T_n, is easily obtained if we note that

$$\frac{dS_1}{dx} = -\frac{dC_1}{dx} = -C_1^2\frac{dT_1}{dx} = -\frac{\eta^2}{2y}$$

When applying a Newton-Raphson iteration, a note of caution is necessary. Lancaster and Blanchard[5] noticed that, in the vicinity of the minimum energy orbit ($x=0$) and for λ in the range $-1.0 \leq \lambda \leq -0.97$ (close to a 360° transfer angle), the second derivative of the time of the flight function vs x is negative. Under these very specialized circumstances, a different iteration technique would be required.

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References

[1] Escobal, P. R., *Methods of Orbit Determination*, Wiley, New York, 1965.

[2] Herrick, S., *Astrodynamics*, Vol. 1, Van Nostrand, London, 1971.

[3] Battin, R. H., *Astronautical Guidance*, McGraw-Hill, New York, 1964.

[4] Battin, R. H., "A New Solution for Lambert's Problem," *Proceedings of the XIXth International Astronautical Congress*, Vol. 2, Pergamon Press, PWN-Polish Scientific Publishers, 1970, pp. 131-150.

[5] Lancaster, E. R. and Blanchard, R. C., "A Unified Form of Lambert's Theorem," NASA TN D-5368, Sept. 1969.

[6] Oberhettinger, F., "Hypergeometric Functions," *Handbook of Mathematical Functions*, edited by A. Abramowitz and I. A. Stegun, Dover Publications, Inc., New York, 1965, pp. 556-561.

[7] Wall, H. S., *Analytic Theory of Continued Fractions*, Van Nostrand, New York, 1948.

[8] Chrystal, G., *Textbook of Algebra*, Pt. 2, Chap. 34, Dover Publications, New York, 1961.

[9] Gautschi, W., "Computational Aspects of Three-Term Recurrence Relations," *SIAM Review*, Vol. 9, Jan. 1967, pp. 29-30.

[10] D'Amario, L. A. and Synnott, S. P., "A Comparison of Solutions of Kepler's and Lambert's Problems," Instrumentation Laboratory, Cambridge, Mass., Report T-520, Nov. 1969.